INVESTIGATION OF THE SYSTEM OF EQUATIONS FOR A COMPRESSIBLE LAMINAR BOUNDARY LAYER IN THE NEIGHBORHOOD OF A STAGNATION LINE

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In addition to a variety of numerical methods, modern laminar boundary layer theory uses asymptotic methods as an efficient methods as an efficient analytical tool. Asymptotic integrationof the equations makes it possible to obtain closed-form expressions for quantities proportional to friction $\varphi''(0)$ and heat flux $g'(0)$. Preliminary determination of the missing values for the zero derivatives makes it possible to reduce boundary value problems in boundary layer theory to the Cauchy problem, thus greatly simplifying the numerical solution.

The general principles of asymptotic integration of boundary layer equations are outlined in Meksyin's monograph [1], which contains a detailed analysis of incompressible boundary layer problems. By formal integration of the boundary layer equations—as linear inhomogeneous equations in the second derivative of the stream function, the problem of determining friction is reduced to the calculation of a Laplace-type integral by the method of steepest descent. This yields an equation for the unknown with a right-hand side in the form of a divergent series, summable by Euler's formula.

In [2, 3] G. A. Tirskii used asymptotic methods to derive similarity relations between the enthalpy and concentration fields, assuming friction to be known from a numerical solution of the problem.

The question of simultaneously determining friction and heat flux in compressible boundary layer problems was formulated in [4], where, for a definite type of equations, a method of calculating these quantities from a system of finite equations was considered.

The present paper is concerned with the system of equations for a compressible laminar boundary layer in the neighborhood of a stagnation line. The physical properties of the gas are given functions of temperature.

A system of equations for friction $\varphi''(0)$ and heat flux $g'(0)$ is obtained, the accuracy of these quantities being independent of the value of the blowing-BLC parameter. Since the numberical solution of the initial system of equations is known for a limited number of parameters of the problem, the analytical formulas derived make it possible to study the effect of each parameter individually. The accuracy with which the unknowns are determined can be established by evaluating subsequent terms of the asymptotic expansions.

A comparison with a numerical solution for some parameters of the problem is given in the last part of the paper.

To the flow of a compressible gas past a plane stagnation point there corresponds the the following boundary value problem [2]:

$$(l\varphi'')' + \varphi\varphi'' = \varphi'^2 - g_0 - (1 - g_0)\, g \qquad \left(g = \frac{h - h_0}{h_\infty - h_0}\right) \tag{1}$$

$$(lg')' + \sigma\varphi g' = 0 \tag{2}$$

$$\varphi(0) = \alpha, \qquad \varphi'(0) = 0, \qquad g(0) = 0, \qquad \varphi'(\infty) = g(\infty) = 1, \tag{3}$$

where the function φ is proportional to the normal velocity component in the boundary layer; g is dimensionless enthalpy, the subscript 0 corresponding to the conditions at the body, the function l is proportional to the compressibility factor $\rho\mu/\rho_0\mu_0$ in which expression ρ is density and μ is the viscosity of the gas, σ stands for the Prandtl number.

In [2] it was shown that the function l can be expressed as follows;

$$l(g) = \frac{(1 + l_1 g)^{1/2}}{1 + l_1 l_2 g}, \qquad l_1 = \frac{1}{g_0} - 1,$$

$$l_2 = \frac{1}{1 + S_{j0}\, g_0^{-1}} \qquad \left(g_0 = \frac{h_0}{h_\infty},\ S_{j0} = \frac{S_j c_{p0}}{T_\infty c_{p\infty}}\right), \tag{4}$$

where S_j is the Sutherland constant. As in [4], we introduce the function

$$R = \frac{1}{l}\left(\varphi + l' + \frac{g_0 + (1 - g_0)\, g - \varphi'^2}{\varphi''}\right). \tag{5}$$

Then equation [1] can be written in a form analogous to the Blasius equation:

$$\varphi''' + R\varphi'' = 0. \tag{6}$$

Double integration of this equation with consideration of the boundary conditions yields the following expression for friction:

$$\varphi''(0) = \tau = \omega^{-1}(\infty), \qquad \omega_{(\infty)} = \int_0^\infty \exp\left(-\int_0^\eta R\, dt\right) d\eta. \tag{7}$$

Considering only the first two terms of the expansion of function R in a Taylor series in the neighborhood of $\eta = 0$, we obtain the principal term of the asymptotic expansion of the quantity $\omega(\infty)$. As shown in [4], for equations of type (1) this term gives a good approximation of the exact value of $\omega(\infty)$.

After transformations, Eq. (7), whose right side contains the principal term of the expansion of $\omega(\infty)$, can be written in the form:

$$\frac{2\tau^2}{g_0 + \alpha\tau + (dl/dg)_0 \tau g_0'} = G(x) \tag{8}$$

$$G(x) = \frac{2}{\sqrt{\pi}} \frac{1}{x} \frac{e^{-x^2}}{1 - \Phi(x)}, \qquad x = \frac{g_0 + \alpha\tau + (dl/dg)_0\, \tau g_0'}{\sqrt{2\,[p - q g_0'\tau + r g_0'^2 \tau^2]}} \tag{9}$$

$$p = g_0(g_0 + \alpha\tau), \qquad q = g_0 - 1 + \alpha(1 + \sigma)\left(\frac{dl}{dg}\right)_0 \tau,$$

$$r = \left(\frac{d^2 l}{dg^2}\right)_0 - 2\left(\frac{dl}{dg}\right)_0^2, \quad \left(\frac{dl}{dg}\right)_0 = \frac{l_1}{2}(1 - 2l_2),$$

$$\left(\frac{d^2 l}{dg^2}\right)_0 = -\frac{l_1^2}{2}\left[l_2 + (1 - 2l_2)\left(2l_2 + \frac{1}{2}\right)\right].$$

The coefficients p, q, r in formulas (9) depend on the parameters of the problem and on friction.

Let us return to the integration of Eq. (2). Introducing the coordinate

$$\eta_1 = \int_0^\eta l^{-1}\, d\eta \tag{10}$$

we obtain Eq. (2) in the following form

$$\frac{d^2 g}{d\eta_1^2} + \sigma\varphi_1 \frac{dg}{d\eta_1} = 0, \qquad \varphi_1 \equiv \varphi[\eta(\eta_1)]. \tag{11}$$

Integrating, with consideration of the boundary conditions, we obtain the following expression for g_0':

$$g_0' = \frac{1}{\omega_1(\infty)}, \qquad \omega_1(\infty) = \int_0^\infty \exp\left(-\int_0^{\eta_1} \varphi_1\, dt\right) d\eta_1. \tag{12}$$

Table 1

k	$\psi_0(k)$	$\psi_1(k)$	k	$\psi_0(k)$	$\psi_1(k)$
0	0.893	0.4514	1.1	1.6812	1.0921
0.1	0.9398	0.4862	1.2	1.7953	1.1921
0.2	0.9903	0.5244	1.3	1.9200	1.3028
0.3	1.0448	0.5662	1.4	2.0563	1.4256
0.4	1.1037	0.6121	1.5	2.2056	1.5619
0.5	1.1674	0.6625	1.6	2.3692	1.7134
0.6	1.2370	0.7179	1.7	2.5488	1.8812
0.7	1.3111	0.7788	1.8	2.7462	2.0696
0.8	1.3923	0.8460	1.9	2.9635	2.2790
0.9	1.4806	0.9200	2	3.2028	2.5127
1	1.5766	1.0017	2.1	3.4669	2.7738

Table 2

Friction (τ)

$\varphi(0)=\alpha$	$\sigma=1$			$\sigma=0.7$		
	$l=1$	$S_{j0}=0.2$	$S_{j0}=0.02$	$l=1$	$S_{j0}=0.2$	$S_{j0}=0.02$
0	0.9629 (0.9548)	0.9117 —	0.8802 —	0.9528 (0.9362)	0.9083 (0.9109)	0.8788 (0.8894)
—0.3	0.7770	0.7383	0.7084	0.7737	0.7388	0.7114
—0.5	0.6661 (0.6669)	0.6373 —	0.6133 —	0.6668 (0.6610)	0.6395 (0.6430)	0.6168 (0.6276)
—1.0	0.4489 (0.4519)	0.4409 —	0.4333 —	0.4540 (0.4551)	0.4441 (0.4458)	0.4349 (0.4380)

Heat Flux (g_0^*)

$\varphi(0)=\alpha$	$\sigma=1$			$\sigma=0.7$		
	$l=1$	$S_{j0}=0.02$	$S_{j0}=0.02$	$l=1$	$S_{j0}=0.2$	$S_{j0}=0.02$
0	0.5540 (0.5421)	0.5327 —	0.5333 —	0.4841 (0.4696)	0.4676 (0.4484)	0.4692 (0.4317)
—0.3	0.3673	0.3470	0.3369	0.3470	0.3302	0.3232
—0.5	0.2623 (0.2580)	0.2453 —	0.2340 —	0.2663 (0.2600)	0.2511 (0.2430)	0.2420 (0.2295)
—1.0	0.0823 (0.0823)	0.0769 —	0.0722 —	0.1120 (0.1112)	0.1044 (0.1010)	0.0983 (0.0932)

In the neighborhood of $\eta = 0$, we have the expansion

$$\int\limits_0^{\eta_1} \varphi_1 dt = \alpha\eta_1 + \frac{\tau}{6}\eta_1^3 + Z(\eta_1), \qquad \tau = \varphi''_{\eta\eta}(0) = \varphi''_{\eta_1\eta_1}(0),$$

$$Z(\eta_1) = \eta_1^4 \sum_{n=0}^{\infty} \frac{\varphi_{\eta_1}^{(n+3)}(0)}{(n+4)!}\eta_1^n. \tag{13}$$

The integrand in the expression $\omega(\infty)$ decreases exponentially, so that the first terms of the expansion will make the principal contribution. Substituting (13) into (12) and expanding $\exp[-Z(\eta_1)]$ in series, after integration, we obtain the expression

$$g_0' \sim \left(\frac{\sigma\tau}{6}\right)^{1/3} \psi_0^{-1}(k)\left[1 + \sum_{m=0}^{\infty} \gamma_m \psi_{m+4}(k)\right]^{-1}, \tag{14}$$

$$k = |\alpha|\left(\frac{\sigma\tau}{6}\right)^{-1/3}\sigma, \qquad \gamma_m = d_m\left(\frac{\sigma\tau}{6}\right)^{-(m+4)/3}\psi_0^{-1}(k),$$

$$d_m = -\sigma\frac{\varphi_{\eta_1}^{m+3}(0)}{(m+4)!}, \quad m = 0, 1, 2, 3. \tag{15}$$

The functions

$$\psi_p(k) = \int\limits_0^{\infty} e^{kx-x^3} x^p\, dx \tag{16}$$

can be expressed in terms of the functions $\varphi_0(k)$ and $\varphi_1(k)$ using the recurence relations

$$\psi_2(k) = {}^1/_3\,[1 + k\psi_0(k)]$$
$$\psi_{p+3}(k) = {}^1/_3\,[(p+1)\psi_p(k) + k\psi_{p+1}(k)] \qquad (p = 0, 1, \ldots). \tag{17}$$

Table 1 gives values of the functions $\varphi_0(k)$ and $\varphi_1(k)$ for $0 \le k \le 2.1$. For values of k not listed in Table 1, it is convenient to use the interpolation formula

$$\psi_p(k+\delta) = \sum_{r=0}^{\infty} \frac{1}{r!}\psi_{p+r}(k)\,\delta^r = \psi_p(k) + \tag{18}$$
$$+ \psi_{p+1}(k)\,\delta + \psi_{p+2}(k)\frac{\delta^2}{2} + \ldots$$

For values of $k < 1$, the functions $\varphi_0(k)$ and $\varphi_1(k)$ can be obtained from asymptotic expressions of the form

$$\psi_0(k) = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)\sum_{n=0}^{\infty}\frac{k^n}{n!}\frac{\Gamma({}^1/_3(n+1))}{\Gamma({}^1/_3)},$$
$$\psi_1(k) = \frac{1}{3}\Gamma\left(\frac{2}{3}\right)\sum_{n=0}^{\infty}\frac{k^n}{n!}\frac{\Gamma({}^1/_3(n+2))}{\Gamma({}^2/_3)}. \tag{19}$$

For example, for $k = 0.6$, three terms of expansion (19) yield a value for the function $\varphi_0(k)$ that differs by less than 0.01% from the tabulated value.

These calculations show that the quantity g_0' can be determined with sufficient accuracy by considering three terms of the expansion in Eq. (14):

$$g_0' = ({}^1/_6\sigma\tau)^{1/3}\psi_0^{-1}(k)\,[1 + \gamma_0\psi_4 + \gamma_1\psi_5]^{-1}. \tag{20}$$

Expressing the coefficients γ_0 and γ_1 explicitly in terms of τ and g_0', and substituting the initial parameters of the variant, we obtain the final equation in the form

$$g_0' = ({}^1/_6\sigma\tau)^{1/3}\psi_0^{-1}(k)\,[1 + A + Bg_0' - Cg_0'^2]^{-1}, \tag{21}$$

where

$$A = \frac{\sigma}{24}\left(\frac{\sigma\tau}{6}\right)^{-4/3} a\left[\frac{\psi_4}{\psi_0} + \frac{1}{5}\frac{\psi_5}{\psi_0}\left(\frac{\sigma\tau}{6}\right)^{-1/3}(-\alpha)\right], \qquad a = g_0 + \alpha\tau,$$

$$B = \frac{\sigma}{24}\left(\frac{\sigma\tau}{6}\right)^{-4/3}\left[b\frac{\psi_4}{\psi_0} + \frac{1}{5}\frac{\psi_5}{\psi_0}\left(\frac{\sigma\tau}{6}\right)^{-1/3} b^*\right], \qquad b = -2\tau\left(\frac{dl}{dg}\right)_0,$$

$$C = \frac{\sigma}{24}\left(\frac{\sigma\tau}{6}\right)^{-4/3}\frac{1}{5}\left(\frac{\sigma\tau}{6}\right)^{-1/3}\frac{\psi_5}{\psi_0}c, \qquad c = \left[4\left(\frac{d^2l}{dg^2}\right)_0 - \left(\frac{dl}{dg}\right)_0^2\right]\tau,$$

$$b^* = 1 - g_0\left[1 - 4\left(\frac{dl}{dg}\right)_0\right] + 3\left(\frac{dl}{dg}\right)_0 \alpha(1+\sigma)\tau. \tag{22}$$

Since the quantity $Bg_0' + Cg_0'^2 \ll 1 + A$, Eq. (21) can be readily solved by successive approximations.

Table 2 gives values of τ and g_0' obtained by solving the system of equations (8), (21) for various combinations of the parameters σ, α, S_{j_0}. The parameter g_0 is equal to 0.5. Values of τ and g_0' obtained by numerical integration are given in parentheses.

Analysis of the data in Table 2 shows that the approximate equations (8), (21) give satisfactory accuracy in calculating the unknowns τ and g_0'.

It is of interest to note that the error involved in the determination of τ and g_0' is roughly constant over the entire range of variation of the parameter α. For a more exact determination of values of g_0' at small α use can be made of asymptotic expansions analogous to those derived in [4]. The same reference also gives a comparison with the result of a numerical integration for $g_0 = 1$.

In conclusion, we note that since the dependence of the solution of the system (8), (21) on the variability of the properties of the gas is described by the quantity $(dl/dg)_0$ (see (22)), it is evidently this parameter which should be considered characteristic in taking into account the compressibility, rather than the factor $l_e = \rho_\infty\mu_\infty/\rho_*\mu_*$, commonly used in the literature for approximating the values of τ and g_0'.

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REFERENCES

1. D. Meksyn, "New methods in laminar boundary-layer theory," Oxford, P. P., 1961.

2. G. A. Tirskii, "Sublimation of a blunt body in the neighborhood of a stagnation point in a plane and axisymmetric gas mixture flow," Zh. vychisl. mekh. i matem. fiz., 1961.

3. G. A. Tirskii, "Ablation of a body in the neighborhood of a stagnation point and line in a dissociated air stream with evaporation of the ablative film, "PMTF, no. 5, 1961

4. B. I. Renznikov and Yu. N. Smyslov, "A method for determining friction and heat flux in self-similar boundary-layer problems," PMTF, no. 1, 1964.

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